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CONVERGENCE OF SERIES IN MITTAG-LEFFLER FUNCTIONS

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Abstract

In this paper Cauchy-Hadamard, Abel, Tauber and Littlewood type theorems for series in Mittag-Leffler functions are given. There are also provided asymptotic formulae for "large" values of indices of these functions, used in the proofs of the convergence theorems for the considered series.

Key words and phrases: Mittag-Leffler functions, Cauchy-Hadamard, Abel, Tauber and Littlewood type theorems, summation of divergent series, asymptotic formula

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1. Introduction. The Mittag–Leffler (M–L) functions E_α (Mittag–Leffler, 1902–1905) and $E_{\alpha,\beta}$ defined in the whole complex plane $\mathbb{C}$ by the power series

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)},\quad E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad \alpha>0,\ \beta>0, \tag{1.1}$$

are natural extensions of the exponential function and trigonometric functions like cos-function. A description of their basic properties appeared yet in the Bateman Project [1] (1953–1954), "Higher Transcendental Functions" (Vol. 3), in a chapter

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devoted to "miscellaneous functions". The functions (1.1) have been studied in detail by Dzrbashjan [2]. The detailed properties of these functions can be found in the contemporary monographs of Kilbas et al. [3] and Podlubny [4]. The M–L functions (1.1) have been used as generating functions of the so-called Gel'fond-Leont'ev operators of generalized integration and differentiation (being operators of fractional calculus) by Dimovski [5] and Dimovski and Kiryakova [6].

In our previous papers (Paneva–Konovska [7–9]) we studied series in systems of some other representatives of Special function of Fractional calculus, which are fractional indices analogues of the Bessel functions and also multi-index Mittag–Leffler functions (in the sense of [10, 11]). There are proved Cauchy-Hadamard, Abel and Tauber type theorems in the complex domain.

In this paper we present some asymptotic formulae for "large" values of indices of the M–L functions (1.1) and study the convergence of series in such functions. We provide ideas of the proofs of some of the theorems and the other proofs follow the lines of the proofs in our previous papers, for series in Bessel type functions. The details, using the specific properties of the Mittag–Leffler functions will be provided elsewhere.

2. Asymptotic formulae. First we give some asymptotic formulae for the Mittag-Leffler functions in the case of "large" values of one of their indices. Denote

$$\theta_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(kn+1)}, \tag{2.1}$$

$$\theta_{n,\beta}(z) = \Gamma(\beta) \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(kn+\beta)}, \tag{2.1.β}$$

$$\theta_{\alpha,n}(z) = \Gamma(n) \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\alpha k+n)}. \tag{2.1.α}$$

Lemma 2.1. *Let $K \subset \mathbb{C}$ be a nonempty compact set. Then there exists a constant C, $0 < C < \infty$, such that for each $n \in \mathbb{N}$ and each $z \in K$ the following inequalities hold*:

$$|\theta_n(z)| \leq C/n!\,, \tag{2.2}$$

$$(2.2.\beta) \qquad |\theta_{n,\beta}(z)| \leq C/(n-1)!\,, \qquad\qquad (2.2.\alpha) \qquad |\theta_{\alpha,n}(z)| \leq C\frac{\Gamma(n)}{\Gamma(\alpha+n)}\,.$$

Proof. First, let $z \in \mathbb{C}$. To prove (2.2.β), we can write

$$\theta_{n,\beta}(z) = \Gamma(\beta) \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(kn+\beta)} = \frac{\Gamma(\beta)}{\Gamma(n+\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(n+\beta)}{\Gamma(kn+\beta)}\, z^k.$$

Denoting

$$\gamma_{n,k} = \frac{\Gamma(n+\beta)}{\Gamma(kn+\beta)}, \quad u_{n,k}(z) = \gamma_{n,k}\, z^k,$$

we obtain that the positive numbers $\gamma_{n,k}$ satisfy the conditions

$$0 < \gamma_{n,k} \leq \frac{1}{(k-1)!}, \text{ for } k \in \mathbb{N}.$$

Then the following estimate holds

$$|u_{n,k}(z)| = \gamma_{n,k}\, |z|^k \leq \frac{|z|^k}{(k-1)!}.$$

Thus, we obtain consecutively

$$|\theta_{n,\beta}(z)| \leq \frac{\Gamma(\beta)}{\Gamma(n+\beta)}\, |z| \, \exp(|z|) = \frac{|z| \, \exp(|z|)}{\prod\limits_{s=0}^{n-1} (\beta+s)},$$

$$|\theta_{n,\beta}(z)| \leq \frac{1}{(n-1)!} \frac{|z| \, \exp(|z|)}{\beta}, \tag{2.3}$$

on $\mathbb{C}$.

Further, for all $z \in K$, the estimate (2.2.β) follows immediately from inequality (2.3).

The proofs for the other two inequalities (2.2) and (2.2.α) go in a similar way, using some inequalities for the Euler Gamma function. ■

By means of these estimates, one can obtain the following

Theorem 2.1. *For the Mittag-Leffler functions* E_n, $E_{n,\beta}$, $E_{\alpha,n}(z)$ $(n \in \mathbb{N})$ *the following asymptotic formulae hold*:

$$E_n(z) = 1 + \theta_n(z), \quad z \in \mathbb{C}, \quad \theta_n(z) \to 0 \quad \text{as} \quad n \to \infty, \tag{2.4}$$

$$E_{n,\beta}(z) = \frac{1}{\Gamma(\beta)}\, (1 + \theta_{n,\beta}(z)), \quad z \in \mathbb{C}, \quad \theta_{n,\beta}(z) \to 0 \quad \text{as} \quad n \to \infty, \tag{2.4.β}$$

$$E_{\alpha,n}(z) = \frac{1}{\Gamma(n)}\, (1 + \theta_{\alpha,n}(z)), \quad z \in \mathbb{C}, \quad \theta_{\alpha,n}(z) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.4.α}$$

The functions $\theta_n(z)$, $\theta_{n,\beta}(z)$, $\theta_{\alpha,n}(z)$ are holomorphic for $z \in \mathbb{C}$. The convergence is uniform on the compact subsets of the complex plane $\mathbb{C}$.

The Proof is obvious.

3. Series in Mittag–Leffler functions. We introduce auxiliary functions, associated with Mittag–Leffler's functions, namely

$$\widetilde{E}_0(z) = 1; \quad \widetilde{E}_n(z) = z^n E_n(z),\ n \in \mathbb{N}, \tag{3.1}$$

$$\widetilde{E}_{0,\beta}(z) = 1; \quad \widetilde{E}_{n,\beta}(z) = \Gamma(\beta) z^n E_{n,\beta}(z),\ n \in \mathbb{N}; \quad \beta > 0, \tag{3.1.β}$$

$$\widetilde{E}_{\alpha,0}(z) = 1; \quad \widetilde{E}_{\alpha,n}(z) = \Gamma(n) z^n E_{\alpha,n}(z),\ n \in \mathbb{N}; \quad \alpha > 0, \tag{3.1.α}$$

and consider series in these functions in the complex plane, respectively of the forms

$$(3.2)\quad \sum_{n=0}^{\infty} a_n \widetilde{E}_n(z), \qquad (3.2.\beta)\quad \sum_{n=0}^{\infty} a_n \widetilde{E}_{n,\beta}(z), \qquad (3.2.\alpha)\quad \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,n}(z),$$

with complex coefficients a_n $(n = 0, 1, 2, \dots)$.

Our main objective is to study the convergence of the series (3.2), (3.2.β), (3.2.α) in the complex plane. Here we propose theorems, corresponding to the classical Cauchy-Hadamard, Abel, Tauber and Littlewood theorems for power series. Such kind of results are provoked by the fact that the solutions of some fractional order differential and integral equations can be written in terms of series (or series of integrals) of Mittag–Leffler functions. Convergence theorems also were obtained for series in other special functions, for example, for series in Laguerre and Hermite polynomials [12–15], and resp. by the author – for series in Bessel functions and their Wright's 2, 3, and 4-index generalizations, see the previous papers [7–9].

4. Cauchy-Hadamard type theorems. First we give a theorem of Cauchy-Hadamard type for each of the above series.

Theorem 4.1 (of Cauchy-Hadamard type)**.** *The domain of convergence of each one of the series* (3.2), (3.2.β), (3.2.α) *with complex coefficients a_n is the disk $|z| < R$ with the radius of convergence $R = 1/\Lambda$, where*

$$\Lambda = \limsup_{n\to\infty} (\ |a_n|\ )^{1/n}. \tag{4.1}$$

The cases $\Lambda = 0$ and $\Lambda = \infty$ can be included in the general case too, provided $1/\Lambda$ means ∞, respectively 0.

5. Abel type theorems. Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angular domain with size $2\varphi < \pi$ and with vertex at the point $z = z_0$, which is symmetric in the straight line defined by the points 0 and z_0.

Theorem 5.1 (of Abel type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, Λ be the real number defined by* (4.1), $0 < \Lambda < \infty$. *Let $K = \{z : z \in \mathbb{C}, |z| < R, R = 1/\Lambda\}$. If $f(z)$, $g(z;\beta)$, $h(z;\alpha)$ are, respectively, the sums of the series* (3.2), (3.2.β), (3.2.α) *on the domain K, and these series converge at the point z_0 of the boundary of K, then*

$$\lim_{z\to z_0} f(z) = \sum_{n=0}^{\infty} a_n \widetilde{E}_n(z_0), \tag{5.1}$$

$$\lim_{z\to z_0} g(z;\ \beta) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{n,\ \beta}(z_0), \tag{5.1.β}$$

$$\lim_{z\to z_0} h(z;\ \alpha) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,\ n}(z_0) \tag{5.1.α}$$

provided $|z| < R$ and $z \in g_\varphi$.

The Proofs of Theorem 4.1 and Theorem 5.1 follow the lines of the analogous type theorems in [7–9].

6. $(\boldsymbol{E}, \boldsymbol{z_0})$ summations. Let us consider the numerical series

$$\sum_{n=0}^{\infty} a_n, \quad a_n \in \mathbb{C}, \quad n = 0, 1, 2, \ldots \tag{6.1}$$

Note that each of the functions $\widetilde{E}_n(z)$, $\widetilde{E}_{n,\beta}(z)$, $\widetilde{E}_{\alpha,n}(z)$ ($n \in \mathbb{N}$), being an entire function, not identically zero, has at most a finite number of zeros in the closed and bounded set $|z| \le R$. Moreover, due to the asymptotic formulae (2.4), (2.4.β), (2.4.α), only finite number of these functions may have some zeros at all.

Let $z_0 \in \mathbb{C}$, $|z_0| = R$, $0 < R < \infty$, $\widetilde{E}_n(z_0) \neq 0, \widetilde{E}_{n,\beta}(z_0) \neq 0$, and $\widetilde{E}_{\alpha,n}(z_0) \neq 0$.

For the sake of brevity, denote

$$E_n^*(z;z_0) = \frac{\widetilde{E}_n(z)}{\widetilde{E}_n(z_0)}, \qquad E_{n,\beta}^*(z;z_0) = \frac{\widetilde{E}_{n,\beta}(z)}{\widetilde{E}_{n,\beta}(z_0)}, \qquad E_{\alpha,n}^*(z;z_0) = \frac{\widetilde{E}_{\alpha,n}(z)}{\widetilde{E}_{\alpha,n}(z_0)}. \tag{6.2}$$

Further, we introduce the following new notion of summability, related to series in M-L functions.

Definition 6.1. *The series* (6.1) *is said to be* (E, z_0)*-summable* (*respectively* $(E_\beta, z_0), (E_\alpha, z_0)$*-summable*) *if the series*

$$\sum_{n=0}^{\infty} a_n E_n^*(z;z_0), \tag{6.3}$$

respectively

$$\sum_{n=0}^{\infty} a_n E^*_{n,\beta}(z; z_0), \tag{6.3.β}$$

$$\sum_{n=0}^{\infty} a_n E^*_{\alpha,n}(z; z_0), \tag{6.3.α}$$

converge in the disk $|z| < R$ *and, moreover, there exists the limit*

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} a_n E^*_{n}(z; z_0), \tag{6.4}$$

respectively

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} a_n E^*_{n,\beta}(z; z_0), \tag{6.4.β}$$

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} a_n E^*_{\alpha,n}(z; z_0), \tag{6.4.α}$$

provided z *remains on the segment* $[0, z_0]$.

Remark 6.1. Every (E, z_0), (E_β, z_0), (E_α, z_0)-summation is regular, and this property is just a particular case of Theorem 5.1.

7. Tauber type theorems. A Tauberian theorem is a statement that relates the Abel summability and the standard convergence of a number series by means of some assumptions imposed on the general term of the series under question. A classical result in this direction is given by Theorem 85 in Hardy [16].

In this paper we extend the validity of such type of assertion to series in Mittag-Leffler functions, by means of the following theorem.

Theorem 7.1 (of Tauber type)**.** *If the series* (6.1) *is* (E, z_0)*-summable, respectively* (E_β, z_0) *and* (E_α, z_0)*-summable, and*

$$\lim_{n\to\infty} n a_n = 0, \tag{7.1}$$

then it is convergent.

Tauber type theorems have been given also for summations by means of Laguerre polynomials [14], and Bessel type functions by the author [7–8].

8. Littlewood type theorems. At first sight, it seems that the condition $a_n = o(1/n)$ is essential. Nevertheless, Littlewood succeeds to weaken it and to obtain the following stronger version of the Tauber theorem (see [16], Theorem 90).

A Littlewood generalization of the $o(n)$-version of the Tauber type theorem (Theorem 7.1) is given below. Similar theorems have been proved in [13] (a generalization of a Tauber type theorem, proven in [14]) and [9].

Theorem 8.1 (of Littlewood type)**.** *If the series* (6.1) *is* (E, z_0)*-summable, respectively* (E_β, z_0) *and* (E_α, z_0)*-summable, and*

$$a_n = O(1/n) \tag{8.1}$$

then the series (6.1) *converges.*

Idea of the Proof. Let z belongs to the segment $[0, z_0]$. Using the asymptotic formula (2.4) for the Mittag–Leffler functions, we obtain

$$a_n E_n^*(z; z_0) = a_n \left(\frac{z}{z_0}\right)^n \frac{1+\theta_n(z)}{1+\theta_n(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \left(1 + \widetilde{\theta}_n(z; z_0)\right),$$

where $\widetilde{\theta}_n(z; z_0) = \dfrac{\theta_n(z) - \theta_n(z_0)}{1+\theta_n(z_0)}$. Then $\widetilde{\theta}_n(z; z_0) = O(1/n!)$, due to (2.2).

Let us write (6.3) in the form

$$\sum_{n=0}^{\infty} a_n E_n^*(z; z_0) = \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \left(1 + \widetilde{\theta}_n(z; z_0)\right). \tag{8.2}$$

Denoting $w_n(z) = a_n \left(\frac{z}{z_0}\right)^n \widetilde{\theta}_n(z; z_0)$, we consider the series $\sum\limits_{n=0}^{\infty} w_n(z)$. Since $|w_n(z)| \le |a_n|\, |\widetilde{\theta}_n(z; z_0)|$and according to condition (8.1) and Lemma 2.1, there exists a constant C, such that $|w_n(z)| \le C/n^2$. Then the series $\sum\limits_{n=0}^{\infty} w_n(z)$ converges, even absolutely and uniformly, on the segment $[0, z_0]$. Therefore (since $\lim\limits_{z\to z_0} w_n(z) = 0$), $\lim\limits_{z\to z_0} \sum\limits_{n=0}^{\infty} w_n(z) = \sum\limits_{n=0}^{\infty} \lim\limits_{z\to z_0} w_n(z) = 0$. Obviously, the assumption that the series (6.1) is (E, z_0) – summable implies the existence of the limit (6.4). Then, we conclude that there exists the limit

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \tag{8.3}$$

and, moreover, $\lim\limits_{z\to z_0} \sum\limits_{n=0}^{\infty} a_n E_n^*(z; z_0) = \lim\limits_{z\to z_0} \sum\limits_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n$. From the existence of the limit (8.3) it follows that the series (6.1) is A-summable. Then according to Theorem 90, the series (6.1) converges.

The other two cases (for (E_β, z_0) and (E_α, z_0) – summations) go analogously. ■

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